

# SUPERCONGRUENCES ON SOME BINOMIAL SUMS INVOLVING LUCAS SEQUENCES

GUO-SHUAI MAO AND HAO PAN

ABSTRACT. In this paper, we confirm several conjectured congruences of Sun concerning the divisibility of binomial sums. For example, with help of a quadratic hypergeometric transformation, we prove that

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k}^2 \frac{P_k}{8^k} \equiv 0 \pmod{p^2}$$

for any prime  $p \equiv 7 \pmod{8}$ , where  $P_k$  is the  $k$ -th Pell number. Further, we also propose three new congruences of the same type.

## 1. INTRODUCTION

In [9], with help of the Gross-Koblitz formula, Mortenson solved the following conjecture of Rodriguez-Villegas [14]:

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \frac{1}{16^k} \equiv \left( \frac{-1}{p} \right) \pmod{p^2}$$

for every odd prime  $p$ , where  $\left( \frac{\cdot}{p} \right)$  denotes the Legendre symbol. Subsequently, the similar congruences were widely studied. For the progress of this topic, the reader may refer to [9, 11, 13, 5, 15, 19, 20, 7, 16, 6, 3, 17]. In [18], Sun proposed many conjectured congruences on the sums of binomial coefficients. Some of those conjectures are of the form

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k}^2 a_n \equiv 0 \pmod{p^2}.$$

For example, Sun conjectured that

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k}^2 \frac{\chi_3(k)}{16^k} \equiv 0 \pmod{p^2}$$

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for any prime  $p \equiv 1 \pmod{12}$ , where  $\chi_3(k)$  equals to the Legendre symbol  $\left(\frac{k}{3}\right)$ .

The main purpose of this paper is to confirm the following conjectures of Sun.

**Theorem 1.1.** *Suppose that  $p$  is a prime.*

(i) *If  $p \equiv 3 \pmod{4}$ , then*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k}^2 \frac{1}{(-8)^k} \equiv 0 \pmod{p^2}. \quad (1.1)$$

(ii) *If  $p \equiv 1 \pmod{12}$ , then*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k}^2 \frac{\chi_3(k)}{16^k} \equiv 0 \pmod{p^2}. \quad (1.2)$$

(iii) *If  $p \equiv 7 \pmod{8}$ , then*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k}^2 \frac{P_k}{8^k} \equiv 0 \pmod{p^2}, \quad (1.3)$$

where the Pell number  $P_k$  is given by

$$P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2.$$

(iv) *If  $p \equiv 11 \pmod{12}$ , then*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{R_k}{(-4)^k} \binom{2k}{k}^2 \equiv 0 \pmod{p^2}, \quad (1.4)$$

where  $R_k$  is given by

$$R_0 = 2, R_1 = 4, R_n = 4R_{n-1} - R_{n-2} \text{ for } n \geq 2.$$

We mention that (1.1), (1.2), (1.3) and (1.4) respectively belong to Conjecture 5.5 of [19] and Conjectures A56, A57, A63 of [18].

The sequences  $\{P_n\}$  and  $\{R_n\}$  in Theorem 1.1 both belong to the second-order linear recurrence sequence. In general, define the Lucas sequences  $\{U_n(a, b)\}$  and  $\{V_n(a, b)\}$  by

$$U_0(a, b) = 0, U_1(a, b) = 1, U_n(a, b) = aU_{n-1}(a, b) - bU_{n-2}(a, b) \text{ for } n \geq 2,$$

and

$$V_0(a, b) = 2, V_1(a, b) = a, V_n(a, b) = aV_{n-1}(a, b) - bV_{n-2}(a, b) \text{ for } n \geq 2.$$

Clearly  $P_n = U_n(2, -1)$  and  $R_n = V_n(4, 1)$ . In fact, it is also easy to see that  $(-2)^{n+1} = V_n(-4, 4)$  and  $\chi_3(n) = U_n(-1, 1)$ . So it is natural to study the arithmetical properties of

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{U_k(a, b)}{16^k} \quad \text{and} \quad \sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{V_k(a, b)}{16^k}.$$

Define the  $n$ -th harmonic number

$$H_n = \sum_{k=1}^n \frac{1}{k}.$$

We have

**Theorem 1.2.** *Suppose that  $p$  is an odd prime and  $a, b \in \mathbb{Z}$ . Then*

$$\begin{aligned} & \sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{V_k(a, b)}{16^k} \\ & \equiv (-1)^{\frac{p-1}{2}} 16^{p-1} \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 V_j(a+2, b+a+1) (1+2pH_{2j}) \pmod{p^2}. \end{aligned} \tag{1.5}$$

Furthermore, if  $p^2$  doesn't divide  $a^2 - 4b$ , then

$$\begin{aligned} & \sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{U_k(a, b)}{16^k} \\ & \equiv (-1)^{\frac{p-1}{2}} 16^{p-1} \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 U_j(a+2, b+a+1) (1+2pH_{2j}) \pmod{p^2}. \end{aligned} \tag{1.6}$$

With help of Theorem 1.2, here we can obtain three new congruences of the same type.

**Theorem 1.3.** *Suppose that  $p$  is a prime.*

(i) *If  $p \equiv 7 \pmod{8}$ , then*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k}^2 \frac{W_k}{4^k} \equiv 0 \pmod{p^2}, \tag{1.7}$$

where  $W_k$  is given by

$$W_0 = 0, \quad W_1 = 1, \quad W_n = 8W_{n-1} + 2W_{n-2} \text{ for } n \geq 2.$$

(ii) If  $p \equiv 1 \pmod{6}$ , then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k}^2 \frac{(-1)^k M_k}{16^k} \equiv 0 \pmod{p^2}, \quad (1.8)$$

where  $M_k$  is given by

$$M_0 = 0, \quad M_1 = 1, \quad M_n = 3M_{n-1} - 3M_{n-2} \text{ for } n \geq 2.$$

(iii) If  $p \equiv 7 \pmod{12}$ , then

$$\sum_{\substack{0 \leq k \leq p-1 \\ k \equiv 0 \pmod{3}}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{1}{16^k} \equiv \frac{1}{3} \sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k}^2 \frac{1}{16^k} \pmod{p^2}. \quad (1.9)$$

First, the proof of Theorem 1.2 will given in the section. It is not difficult to check that  $2^n P_n = U_n(4, -4)$  and  $P_{2n} = U_n(6, 1)$ . Then according to Theorem 1.2, in order to prove (1.3), we only need to show that

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 P_{2j} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 P_{2j} H_{2j} \equiv 0 \pmod{p}.$$

However, as we shall later, the former one is not easy to prove. So in the third section, we shall firstly establish an auxiliary lemma, by using some quadratic hypergeometric transformations. Further, the similar divisible congruences for  $R_n$  and  $W_n$  will be also proved. Finally, in Section 4, we shall conclude the proofs of Theorems 1.1 and 1.3.

## 2. PROOF OF THEOREM 1.2

Below we always assume that  $p$  is an odd prime. In this section, we shall prove

**Theorem 2.1.**

$$\begin{aligned} & \sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{(z-1)^k}{16^k} \\ & \equiv (-1)^{\frac{p-1}{2}} 16^{p-1} \sum_{j=0}^{\frac{p-1}{2}} z^j \binom{\frac{p-1}{2}}{j}^2 (1 + 2pH_{2j}) \pmod{p^2}. \end{aligned} \quad (2.1)$$

**Lemma 2.1.**

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \frac{(1-z)^k}{16^k} \equiv (-1)^{\frac{p-1}{2}} 4^{p-1} \sum_{j=0}^{\frac{p-1}{2}} z^j \binom{\frac{p-1}{2}}{j}^2 (1+pH_{\frac{p-1}{2}-j}) \pmod{p^2}. \quad (2.2)$$

*Proof.* Clearly

$$\begin{aligned} \binom{2k}{k}^2 &= 16^k \binom{-\frac{1}{2}}{k}^2 \equiv \frac{16^k}{(k!)^2} \prod_{j=0}^{k-1} \left( \left( -\frac{1}{2} - j \right)^2 - \frac{p^2}{4} \right) \\ &= \frac{16^k}{(k!)^2} \prod_{j=0}^{k-1} \left( \frac{p-1}{2} - j \right) \left( -\frac{p+1}{2} - j \right) \\ &= 16^k \binom{\frac{p-1}{2}}{k} \binom{-\frac{p+1}{2}}{k} \pmod{p^2}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \frac{(1-z)^k}{16^k} &\equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k} \binom{-\frac{p+1}{2}}{k} \sum_{j=0}^k \binom{k}{j} (-z)^j \\ &= \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} (-z)^j \sum_{k=j}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}-j}{k-j} \binom{-\frac{p+1}{2}}{k} \pmod{p^2}. \end{aligned}$$

In view of the Chu-Vandemonde identity, we have

$$\sum_{k=j}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}-j}{k-j} \binom{-\frac{p+1}{2}}{k} = \binom{-1-j}{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}} \binom{\frac{p-1}{2}+j}{\frac{p-1}{2}}.$$

Since

$$\binom{\frac{p-1}{2}+j}{\frac{p-1}{2}} = \frac{\binom{\frac{p-1}{2}}{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}}{\binom{\frac{p-1}{2}}{\frac{p-1}{2}+j}}$$

and

$$\binom{p-1}{k} = \prod_{i=1}^k \left( \frac{p}{i} - 1 \right) \equiv (-1)^k (1 - pH_k) \pmod{p^2},$$

we obtain that

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \cdot \frac{(1-z)^k}{16^k} &\equiv \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} (-z)^j \cdot (-1)^{\frac{p-1}{2}} \frac{\binom{p-1}{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}}{\binom{p-1}{\frac{p-1}{2}-j}} \\ &\equiv \binom{p-1}{\frac{p-1}{2}} \sum_{j=0}^{\frac{p-1}{2}} z^j \binom{\frac{p-1}{2}}{j}^2 (1 + pH_{\frac{p-1}{2}-j}) \pmod{p^2}. \end{aligned}$$

Using the following congruence of Morley [8],

$$\binom{p-1}{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} 4^{p-1} \pmod{p^2},$$

we get (2.1).  $\square$

**Lemma 2.2.**

$$\begin{aligned} &(-1)^{\frac{p+1}{2}} \sum_{k=0}^{\frac{p-1}{2}} (1-z)^k \binom{2k}{k}^2 \frac{H_k}{16^k} \\ &\equiv \frac{2^{p+1} - 4}{p} \sum_{j=0}^{\frac{p-1}{2}} z^j \binom{\frac{p-1}{2}}{j}^2 + \sum_{j=0}^{\frac{p-1}{2}} z^j \binom{\frac{p-1}{2}}{j}^2 H_j \pmod{p}. \end{aligned} \quad (2.3)$$

*Proof.* Clearly

$$\binom{2k}{k} = (-4)^k \binom{-\frac{1}{2}}{k} \equiv (-4)^k \binom{\frac{p-1}{2}}{k} \pmod{p}.$$

So

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \frac{(1-z)^k H_k}{16^k} &\equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 H_k \sum_{j=0}^k \binom{k}{j} (-z)^j \\ &\equiv \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} (-z)^j \sum_{k=j}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}-j}{k-j} \binom{\frac{p-1}{2}}{k} H_k \pmod{p}. \end{aligned}$$

Apparently

$$\lim_{t \rightarrow 0} \frac{d}{dt} \left( \binom{\frac{p-1}{2}-t}{\frac{p-1}{2}-k} \right) = - \binom{\frac{p-1}{2}}{\frac{p-1}{2}-k} \sum_{i=0}^{\frac{p-1}{2}-k-1} \frac{1}{\frac{p-1}{2}-i} = \binom{\frac{p-1}{2}}{k} (H_k - H_{\frac{p-1}{2}}).$$

Note that

$$\sum_{k=j}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}-j}{k-j} \binom{\frac{p-1}{2}-t}{\frac{p-1}{2}-k} = \binom{p-1-j-t}{\frac{p-1}{2}-j},$$

and

$$\begin{aligned}
 \lim_{t \rightarrow 0} \frac{d}{dt} \left( \binom{p-1-j-t}{\frac{p-1}{2}-j} \right) &= - \binom{p-1-j}{\frac{p-1}{2}-j} \sum_{i=0}^{\frac{p-1}{2}-j-1} \frac{1}{p-1-j-i} \\
 &\equiv \binom{p-1-j}{\frac{p-1}{2}-j} \sum_{i=0}^{\frac{p-1}{2}-j-1} \frac{1}{i+j+1} \\
 &= \binom{p-1-j}{\frac{p-1}{2}-j} (H_{\frac{p-1}{2}} - H_j) \pmod{p}.
 \end{aligned}$$

We obtain that

$$\begin{aligned}
 &\sum_{k=j}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}-j}{k-j} \binom{\frac{p-1}{2}}{k} H_k \\
 &= \sum_{k=j}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}-j}{\frac{p-1}{2}-k} \binom{\frac{p-1}{2}}{k} H_{\frac{p-1}{2}} + \lim_{t \rightarrow 0} \frac{d}{dt} \left( \sum_{k=j}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}-j}{k-j} \binom{\frac{p-1}{2}}{\frac{p-1}{2}-k} \right) \\
 &\equiv \binom{p-1-j}{\frac{p-1}{2}-j} (2H_{\frac{p-1}{2}} - H_j) = (-1)^{\frac{p-1}{2}-j} \binom{-\frac{p+1}{2}}{\frac{p-1}{2}-j} (2H_{\frac{p-1}{2}} - H_j) \\
 &\equiv (-1)^{\frac{p-1}{2}-j} \binom{\frac{p-1}{2}}{\frac{p-1}{2}-j} (2H_{\frac{p-1}{2}} - H_j) \pmod{p}.
 \end{aligned}$$

Thus (2.2) immediately follows from a congruence of Lehmer [4] as follows:

$$H_{\frac{p-1}{2}} \equiv -\frac{2^p - 2}{p} \pmod{p}.$$

□

*Proof of Theorem 2.1.* Note that

$$4^{p-1} = 1 + (2^{p-1} - 1)(2^{p-1} + 1) \equiv 1 + 2(2^{p-1} - 1) \pmod{p^2}.$$

Combining Lemmas 2.1 and 2.2, we obtain that

$$\begin{aligned}
& (-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{(z-1)^k}{16^k} \\
& \equiv (-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \frac{(1-z)^k}{16^k} (1 - pH_k) \\
& \equiv (3 \cdot 2^p - 5) \sum_{j=0}^{\frac{p-1}{2}} z^j \binom{\frac{p-1}{2}}{j}^2 + p \sum_{j=0}^{\frac{p-1}{2}} z^j \binom{\frac{p-1}{2}}{j}^2 (H_j + H_{\frac{p-1}{2}-j}) \pmod{p^2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
H_{\frac{p-1}{2}-j} &= \sum_{i=j+1}^{\frac{p-1}{2}} \frac{1}{\frac{p+1}{2}-i} \equiv - \sum_{i=j+1}^{\frac{p-1}{2}} \frac{2}{2i-1} \\
&= \sum_{i=1}^j \frac{2}{2i-1} - \sum_{i=1}^{\frac{p-1}{2}} \frac{2}{2i-1} = (2H_{2j} - H_j) - (2H_{p-1} - H_{\frac{p-1}{2}}) \\
&\equiv 2H_{2j} - H_j + H_{\frac{p-1}{2}} \pmod{p},
\end{aligned}$$

where in the last step we use the well-known fact

$$H_{p-1} \equiv 0 \pmod{p}.$$

Thus

$$\begin{aligned}
& (-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{(z-1)^k}{16^k} \\
& \equiv (3 \cdot 2^p - 5) \sum_{j=0}^{\frac{p-1}{2}} z^j \binom{\frac{p-1}{2}}{j}^2 + p \sum_{j=0}^{\frac{p-1}{2}} z^j \binom{\frac{p-1}{2}}{j}^2 (2H_{2j} + H_{\frac{p-1}{2}}) \\
& \equiv (2^{p+1} - 3) \sum_{j=0}^{\frac{p-1}{2}} z^j \binom{\frac{p-1}{2}}{j}^2 + 2p \sum_{j=0}^{\frac{p-1}{2}} z^j \binom{\frac{p-1}{2}}{j}^2 H_{2j} \pmod{p^2}.
\end{aligned}$$

Finally, we have

$$16^{p-1} = 1 + (2^{p-1} - 1)(2^{p-1} + 1)(4^{p-1} + 1) \equiv 1 + 4(2^{p-1} - 1) \pmod{p^2}.$$

The proof of (2.1) is concluded.  $\square$

Now Theorem 1.2 is an easy consequence of Theorem 2.1. Let  $\alpha$  and  $\beta$  be the two roots of the equation  $x^2 - ax + b = 0$ . Then  $\alpha + 1$  and



$\beta + 1$  are also the two roots of  $x^2 - (a + 2)x + (b + a + 1) = 0$ . We know that

$$V_n(a, b) = \alpha^n + \beta^n, \quad V_n(a + 2, b + a + 1) = (\alpha + 1)^n + (\beta + 1)^n$$

for each  $n \geq 0$ . And by (2.1),

$$\begin{aligned} & \sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{\alpha^k + \beta^k}{16^k} \\ & \equiv (-1)^{\frac{p-1}{2}} 16^{p-1} \sum_{j=0}^{\frac{p-1}{2}} ((\alpha + 1)^k + (\beta + 1)^k) \binom{\frac{p-1}{2}}{j}^2 (1 + 2pH_{2j}) \pmod{p^2}. \end{aligned}$$

Then (1.5) is derived.

Suppose that  $p^2$  doesn't divide  $a^2 - 4b$ . Then

$$U_n(a, b) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad U_n(a + 2, b + a + 1) = \frac{(\alpha + 1)^n - (\beta + 1)^n}{\alpha - \beta}.$$

Note that  $\alpha - \beta = \pm\sqrt{a^2 - 4b}$ . So

$$\begin{aligned} & \sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{\alpha^k - \beta^k}{16^k(\alpha - \beta)} \\ & \equiv (-1)^{\frac{p-1}{2}} 16^{p-1} \sum_{j=0}^{\frac{p-1}{2}} \frac{(\alpha + 1)^k - (\beta + 1)^k}{\alpha - \beta} \binom{\frac{p-1}{2}}{j}^2 (1 + 2pH_{2j}) \pmod{p^{3/2}}. \end{aligned}$$

Since both sides of the above congruences are factly rational  $p$ -integers, they must also be congruent modulo  $p^2$ , i.e., (1.6) is valid.

### 3. QUADRATIC HYPERGEOMETRIC TRANSFORMATIONS

In this section, we shall use the quadratic hypergeometric transformations to deduce some auxiliary results on  $P_n$  and  $R_n$ , which is necessary for the proof of (1.3) and (1.4). Define the hypergeometric function

$${}_2F_1\left(\begin{matrix} a & b \\ c \end{matrix} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{z^k}{k!},$$

where

$$(a)_k = \begin{cases} a(a+1) \cdots (a+k-1), & \text{if } k \geq 1, \\ 1, & \text{if } k = 0. \end{cases}$$

It is easy to see that if  $n$  is a non-negative integer, then

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{k} z^k = \sum_{k=0}^{\infty} \frac{(-n)_k (-m)_k}{(1)_k} \cdot \frac{z^k}{k!} = {}_2F_1 \left( \begin{matrix} -n & -m \\ 1 \end{matrix} \middle| z \right).$$

**Lemma 3.1.** (i) *Suppose that  $p \equiv 5, 7 \pmod{8}$  is prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (1 + \sqrt{2})^{2k} \equiv 0 \pmod{p}. \quad (3.1)$$

(ii) *Suppose that  $p \equiv 2 \pmod{3}$  is a prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^k (2 + \sqrt{3})^{2k} \equiv 0 \pmod{p}. \quad (3.2)$$

(iii) *Suppose that  $p \equiv 3 \pmod{4}$  is a prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (3 + 2\sqrt{2})^{2k} \equiv 0 \pmod{p}. \quad (3.3)$$

*Proof.* (i) We need the following quadratic transformation for the hypergeometric functions [1, Theorem 3.1.1]:

$${}_2F_1 \left( \begin{matrix} a & b \\ a - b + 1 \end{matrix} \middle| z \right) = (1 - z)^{-a} {}_2F_1 \left( \begin{matrix} \frac{1}{2}a & \frac{1}{2}a - b + \frac{1}{2} \\ a - b + 1 \end{matrix} \middle| -\frac{4z}{(1 - z)^2} \right), \quad (3.4)$$

where  $|z| < 1$ . Let  $z = (1 + \sqrt{2})^2$ . It is easy to check that

$$-\frac{4z}{(1 - z)^2} = -1.$$

Suppose that  $p \equiv 5 \pmod{8}$ . Substituting  $a = b = -\frac{p-1}{2}$  in (3.4), we get

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 z^k &= {}_2F_1 \left( \begin{matrix} -\frac{p-1}{2} & -\frac{p-1}{2} \\ 1 \end{matrix} \middle| z \right) \\ &= (1 - z)^{\frac{p-1}{2}} {}_2F_1 \left( \begin{matrix} -\frac{p-1}{4} & \frac{p+1}{4} \\ 1 \end{matrix} \middle| -1 \right). \end{aligned}$$

Notice that both  ${}_2F_1 \left( \begin{matrix} -\frac{p-1}{2} & -\frac{p-1}{2} \\ 1 \end{matrix} \middle| z \right)$  and  ${}_2F_1 \left( \begin{matrix} -\frac{p-1}{4} & \frac{p+1}{4} \\ 1 \end{matrix} \middle| -1 \right)$  are factly the finite summations. So the requirement  $|z| < 1$  can be ignored

here. Then (3.1) follows from that

$$\begin{aligned} {}_2F_1\left(-\frac{p-1}{4} \quad \frac{p+1}{4} \middle| -1\right) &= \sum_{k=0}^{\frac{p-1}{4}} \binom{\frac{p-1}{4}}{k} \binom{-\frac{p+1}{4}}{k} (-1)^k \\ &\equiv \sum_{k=0}^{\frac{p-1}{4}} \binom{\frac{p-1}{4}}{k}^2 (-1)^k = 0 \pmod{p}, \end{aligned}$$

where in the last step we use the well-known fact [2, (3.32)] that

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^2 \equiv \begin{cases} (-1)^{\frac{n}{2}} \binom{n}{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \quad (3.5)$$

Suppose that  $p \equiv 7 \pmod{8}$ . By the Lucas theorem, we have

$$\binom{\frac{3p-1}{2}}{k} \equiv \binom{\frac{3p-1}{2}}{p+k} \equiv \binom{\frac{p-1}{2}}{k} \pmod{p}$$

for  $0 \leq k \leq \frac{p-1}{2}$ . So

$$\sum_{k=0}^{\frac{3p-1}{2}} \binom{\frac{3p-1}{2}}{k}^2 z^k \equiv (1+z^p) \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 z^k \pmod{p}.$$

Note that

$$1+z^p \equiv (1+z)^p = (4+2\sqrt{2})^p \pmod{p}.$$

Then  $1+z^p$  is prime to  $p$  since  $4+2\sqrt{2}$  is prime to  $p$ . Using (3.4) with  $a=b=-\frac{3p-1}{2}$ , we obtain that

$$\begin{aligned} \sum_{k=0}^{\frac{3p-1}{2}} \binom{\frac{3p-1}{2}}{k}^2 z^k &= {}_2F_1\left(-\frac{3p-1}{2} \quad -\frac{3p-1}{2} \middle| z\right) \\ &= (1-z)^{\frac{3p-1}{2}} {}_2F_1\left(-\frac{3p-1}{4} \quad \frac{3p+1}{4} \middle| -1\right). \end{aligned}$$

In view of (3.5), we have

$$\begin{aligned} {}_2F_1\left(-\frac{3p-1}{4} \quad \frac{3p+1}{4} \middle| -1\right) &= \sum_{k=0}^{\frac{3p-1}{4}} \binom{\frac{3p-1}{4}}{k} \binom{-\frac{3p+1}{4}}{k} (-1)^k \\ &\equiv \sum_{k=0}^{\frac{3p-1}{4}} \binom{\frac{3p-1}{4}}{k}^2 (-1)^k = 0 \pmod{p}. \end{aligned}$$

So (3.1) is also valid when  $p \equiv 7 \pmod{8}$ .

(ii) We shall use another quadratic transformation as follows [1, (3.1.9)]:

$${}_2F_1\left(\begin{matrix} a & b \\ a-b+1 \end{matrix} \middle| z\right) = (1+z)^{-a} {}_2F_1\left(\begin{matrix} \frac{1}{2}a & \frac{1}{2}a + \frac{1}{2} \\ a-b+1 \end{matrix} \middle| \frac{4z}{(1+z)^2}\right). \quad (3.6)$$

Let  $z = -(2 + \sqrt{3})^2$ . Then we have

$$\frac{4z}{(1+z)^2} = -\frac{1}{3}.$$

Applying (3.6) with  $a = b = -\frac{p-1}{2}$ , we get that

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 z^k = {}_2F_1\left(\begin{matrix} -\frac{p-1}{2} & -\frac{p-1}{2} \\ 1 \end{matrix} \middle| z\right) = (1+z)^{\frac{p-1}{2}} {}_2F_1\left(\begin{matrix} -\frac{p-1}{4} & -\frac{p-3}{4} \\ 1 \end{matrix} \middle| -\frac{1}{3}\right).$$

It suffices to show that

$${}_2F_1\left(\begin{matrix} -\frac{p-1}{4} & -\frac{p-3}{4} \\ 1 \end{matrix} \middle| -\frac{1}{3}\right) \equiv 0 \pmod{p}$$

when  $p \equiv 2 \pmod{3}$ . Note that

$$\begin{aligned} {}_2F_1\left(\begin{matrix} -\frac{p-1}{4} & -\frac{p-3}{4} \\ 1 \end{matrix} \middle| -\frac{1}{3}\right) &= \sum_{k=0}^{\frac{p-3}{4}} \frac{(-\frac{p-1}{4})_k (-\frac{p-3}{4})_k}{(1)_k k!} \cdot \left(-\frac{1}{3}\right)^k \\ &\equiv \sum_{k=0}^{\frac{p-3}{4}} \frac{(-\frac{p-1}{4})_k (-\frac{p-3}{4})_k}{\left(\frac{p}{2} + 1\right)_k k!} \cdot \left(-\frac{1}{3}\right)^k \\ &= {}_2F_1\left(\begin{matrix} -\frac{p-1}{4} & -\frac{p-1}{4} + \frac{1}{2} \\ \frac{p}{2} + 1 \end{matrix} \middle| -\frac{1}{3}\right) \pmod{p}. \end{aligned}$$

It is known [12, 15.4.31] that

$${}_2F_1\left(\begin{matrix} a & a + \frac{1}{2} \\ \frac{3}{2} - 2a \end{matrix} \middle| -\frac{1}{3}\right) = \left(\frac{8}{9}\right)^{-2a} \cdot \frac{\Gamma(\frac{4}{3})\Gamma(\frac{3}{2} - 2a)}{\Gamma(\frac{3}{2})\Gamma(\frac{4}{3} - 2a)}.$$

Thus

$${}_2F_1\left(\begin{matrix} -\frac{p-1}{4} & -\frac{p-1}{4} + \frac{1}{2} \\ \frac{p}{2} + 1 \end{matrix} \middle| -\frac{1}{3}\right) = \left(\frac{8}{9}\right)^{\frac{p-1}{2}} \cdot \frac{\Gamma(\frac{4}{3})\Gamma(\frac{3}{2} + \frac{p-1}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{4}{3} + \frac{p-1}{2})}.$$

When  $p \equiv 2 \pmod{3}$ ,  $3j+4 \neq p$  for any  $0 \leq j \leq \frac{p-1}{2}-1$ . But  $2j+3 = p$  if  $j = \frac{p-1}{2} - 1$ . So for prime  $p \equiv 2 \pmod{3}$ , we always have

$${}_2F_1\left(\begin{matrix} -\frac{p-1}{4} & -\frac{p-3}{4} \\ \frac{p}{2} + 1 \end{matrix} \middle| -\frac{1}{3}\right) = \frac{8^{\frac{p-1}{2}}}{9^{\frac{p-1}{2}}} \prod_{j=0}^{\frac{p-1}{2}-1} \frac{\frac{3}{2} + j}{\frac{4}{3} + j} \equiv 0 \pmod{p}.$$

(iii) According to [1, (3.1.11)], we have

$${}_2F_1\left(\begin{matrix} a & b \\ a-b+1 \end{matrix} \middle| z^2\right) = (1+z)^{-2a} {}_2F_1\left(\begin{matrix} a & a-b+\frac{1}{2} \\ 2a-2b+1 \end{matrix} \middle| \frac{4z}{(1+z)^2}\right), \quad (3.7)$$

Let  $z = -(3 + 2\sqrt{2})$ . Apparently

$$\frac{4z}{(1+z)^2} = -1.$$

It follows from (3.7) that

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 z^{2k} &= {}_2F_1\left(\begin{matrix} -\frac{p-1}{2} & -\frac{p-1}{2} \\ 1 \end{matrix} \middle| z^2\right) \\ &= (1+z)^{p-1} {}_2F_1\left(\begin{matrix} -\frac{p-1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle| -1\right). \end{aligned}$$

If  $p \equiv 3 \pmod{4}$ , then

$$\begin{aligned} {}_2F_1\left(\begin{matrix} -\frac{p-1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle| -1\right) &= \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k} \binom{-\frac{1}{2}}{k} (-1)^k \\ &\equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^k \equiv 0 \pmod{p}. \end{aligned}$$

Thus (3.3) is also confirmed.  $\square$

**Lemma 3.2.** *Suppose that  $p$  is a prime.*

(i) *If  $p \equiv 7 \pmod{8}$ , then*

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 P_{2k} \equiv 0 \pmod{p^2}. \quad (3.8)$$

(ii) *If  $p \equiv 11 \pmod{12}$ , then*

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^k R_{2k} \equiv 0 \pmod{p^2} \quad (3.9)$$

(iii) *If  $p \equiv 7 \pmod{8}$ , then*

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \frac{W_{2k}}{2^k} \equiv 0 \pmod{p^2} \quad (3.10)$$

*Proof.* (i) Let  $\alpha = 1 + \sqrt{2}$  and  $\beta = 1 - \sqrt{2}$ . We know that

$$P_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}.$$

Clearly

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \beta^{2k} = \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \beta^{p-1-2k} = \beta^{p-1} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \alpha^{2k}.$$

So

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (\alpha^{2k} - \beta^{2k}) = (1 - \beta^{p-1}) \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \alpha^{2k}.$$

If  $p \equiv 7 \pmod{8}$ , then

$$2^{\frac{p-1}{2}} \equiv 1 \pmod{p}.$$

It follows that

$$\beta^{p-1} = \frac{(1 - \sqrt{2})^p}{1 - \sqrt{2}} \equiv \frac{1 - 2^{\frac{p-1}{2}} \cdot \sqrt{2}}{1 - \sqrt{2}} \equiv \frac{1 - \sqrt{2}}{1 - \sqrt{2}} \equiv 1 \pmod{p}. \quad (3.11)$$

Similarly, we also have  $\alpha^{p-1} \equiv 1 \pmod{p}$ . In view of (3.1) and (3.11), when  $p \equiv 7 \pmod{8}$ , we can get

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 P_{2k} &= \frac{1}{\alpha - \beta} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (\alpha^{2k} - \beta^{2k}) \\ &= \frac{1 - \beta^{p-1}}{\alpha - \beta} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \alpha^{2k} \equiv 0 \pmod{p^2}. \end{aligned}$$

(ii) Let  $\alpha = 2 + \sqrt{3}$  and  $\beta = 2 - \sqrt{3}$ . Then

$$R_k = \alpha^k + \beta^k.$$

Suppose that  $p \equiv 11 \pmod{12}$ . By the quadratic reciprocity theorem,

$$\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{3-1}{2}} \left(\frac{p}{3}\right) = (-1) \cdot (-1) = 1.$$

So  $3^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ . Similarly as (3.11), we can get

$$\alpha^{p-1} \equiv \beta^{p-1} \equiv 1 \pmod{p}.$$

Furthermore,

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^k \beta^{2k} &= \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^{\frac{p-1}{2}-k} \beta^{p-1-2k} \\ &= -\beta^{p-1} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^k \alpha^{2k}. \end{aligned}$$

It follows from (3.2) that

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^k (\alpha^{2k} + \beta^{2k}) &= (1 - \beta^{p-1}) \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^k \alpha^{2k} \\ &\equiv 0 \pmod{p^2}. \end{aligned}$$

(iii) Let  $\alpha = 3 + 2\sqrt{2}$  and  $\beta = 3 - 2\sqrt{2}$ . It is easy to verify that

$$W_{2k} = 2^{k+2} \cdot \frac{\alpha^{2k} - \beta^{2k}}{\alpha^2 - \beta^2} \quad (3.12)$$

for each  $k \geq 0$ . If  $p \equiv 7 \pmod{8}$ , then we also have  $\alpha^{p-1} \equiv \beta^{p-1} \equiv 1 \pmod{p}$ . Similarly, we also can get

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \frac{W_{2k}}{2^k} &= \frac{4}{\alpha^2 - \beta^2} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (\alpha^{2k} - \beta^{2k}) \\ &= \frac{1 - \beta^{p-1}}{6\sqrt{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \alpha^{2k} \equiv 0 \pmod{p^2}. \end{aligned}$$

□

#### 4. PROOFS OF THEOREMS 1.1 AND 1.3

We firstly consider (1.1). It is easy to check that  $V_n(-4, 4) = (-2)^{n+1}$  and  $V_n(-2, 1) = 2(-1)^n$ . Substituting  $a = -4$  and  $b = 4$  in (2.1), we get

$$\begin{aligned} &\sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{1}{(-8)^k} \\ &\equiv (-1)^{\frac{p-1}{2}} 16^{p-1} \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j}^2 (1 + 2pH_{2j}) \pmod{p^2}. \end{aligned}$$

Suppose that  $p \equiv 3 \pmod{4}$ . By (3.5),

$$\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j}^2 = 0.$$

And in view of (4.4), we have

$$\begin{aligned} \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j}^2 H_{2j} &= (-1)^{\frac{p-1}{2}} \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j}^2 H_{p-1-2j} \\ &\equiv - \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j}^2 H_{2j} \pmod{p}, \end{aligned}$$

which clearly implies

$$\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j}^2 H_{2j} \equiv 0 \pmod{p}.$$

Thus

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{1}{(-8)^k} \equiv 0 \pmod{p^2}$$

for any prime  $p \equiv 3 \pmod{4}$ .

Let us consider (1.2). We also have  $U_n(-1, 1) = \chi_3(n)$  and  $U_n(1, 1) = (-1)^{n-1} \chi_3(n)$ . Applying Theorem 1.2 with  $a = -1$  and  $b = 1$ , we obtain that

$$\begin{aligned} &\sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{\chi_3(k)}{16^k} \\ &\equiv (-1)^{\frac{p-1}{2}} 16^{p-1} \sum_{j=0}^{\frac{p-1}{2}} (-1)^{j-1} \chi_3(j) \binom{\frac{p-1}{2}}{j}^2 (1 + 2pH_{2j}) \pmod{p^2}. \end{aligned}$$

Suppose that the prime  $p \equiv 1 \pmod{12}$ . Then

$$(-1)^{\frac{p-1}{2}} \chi_3\left(\frac{p-1}{2} - j\right) = \chi_3(-j) = -\chi_3(j).$$

It follows that

$$\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \chi_3(k) \binom{\frac{p-1}{2}}{j}^2 = - \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \chi_3(k) \binom{\frac{p-1}{2}}{j}^2,$$



i.e.,

$$\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \chi_3(k) \binom{\frac{p-1}{2}}{j}^2 = 0.$$

Similarly, we also have

$$\sum_{j=0}^{\frac{p-1}{2}} (-1)^j \chi_3(k) \binom{\frac{p-1}{2}}{j}^2 H_{2j} \equiv 0 \pmod{p}.$$

Thus (1.2) is also proved.

The proofs of (1.8) and (1.9) are very similar as the one of (1.2). Clearly  $(-1)^k M_k = U_k(-3, 3)$ . Suppose that  $p \equiv 1 \pmod{6}$ . By Theorem 1.2, we only need to show that

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 U_k(-1, 1) (1 + 2pH_{2j}) \equiv 0 \pmod{p^2}.$$

Since  $U_k(-1, 1) = \chi_3(k)$  and  $\chi_3(\frac{p-1}{2} - k) = -\chi_3(k)$  now, similarly we also have

$$\sum_{j=0}^{\frac{p-1}{2}} \chi_3(k) \binom{\frac{p-1}{2}}{j}^2 = 0 \quad \text{and} \quad \sum_{j=0}^{\frac{p-1}{2}} \chi_3(k) \binom{\frac{p-1}{2}}{j}^2 H_{2j} \equiv 0 \pmod{p}.$$

Then (1.8) is derived. Let

$$\delta_3(k) = \begin{cases} 2, & \text{if } k \equiv 0 \pmod{3}, \\ -1, & \text{otherwise.} \end{cases}$$

Obviously (1.9) is equivalent to

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \frac{\delta_3(n)}{16^k} \equiv 0 \pmod{p^2}$$

for each prime  $p \equiv 7 \pmod{12}$ . Since  $\delta_3(k) = V_k(-1, 1)$ , it suffices to prove that

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 V_k(1, 1) (1 + 2pH_{2j}) \equiv 0 \pmod{p^2}.$$

It is easy to verify

$$V_k(1, 1) = -V_{6h+3-k}(1, 1)$$

for any  $h \geq 0$  and  $0 \leq k \leq 6h + 3$ . So if  $\frac{p-1}{2} \equiv 3 \pmod{6}$ , then

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 V_j(1, 1) = 0 \quad \text{and} \quad \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 V_j(1, 1) H_{2j} \equiv 0 \pmod{p}.$$

Finally, let us turn to (1.3), (1.4) and (1.7). We require some additional auxiliary results.

**Lemma 4.1.** (i) Suppose that  $p \equiv \pm 1 \pmod{8}$  is a prime. Then

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 P_{2k} H_{2k} \equiv 0 \pmod{p}. \quad (4.1)$$

(ii) Suppose that  $p \equiv 11 \pmod{12}$  is a prime. Then

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^k R_{2k} H_{2k} \equiv 0 \pmod{p}. \quad (4.2)$$

(iii) Suppose that  $p \equiv 7 \pmod{8}$  is a prime. Then

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \frac{W_{2k}}{2^k} \cdot H_{2k} \equiv 0 \pmod{p}. \quad (4.3)$$

*Proof.* (i) Let  $\alpha = 1 + \sqrt{2}$  and  $\beta = 1 - \sqrt{2}$ . Clearly

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (\alpha^{2k} - \beta^{2k}) H_{2k} &= \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (\alpha^{p-1-2k} - \beta^{p-1-2k}) H_{p-1-2k} \\ &= \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (\alpha^{p-1} \beta^{2k} - \beta^{p-1} \alpha^{2k}) H_{p-1-2k}. \end{aligned}$$

Note that for any  $1 \leq j \leq p-2$ ,

$$H_{p-1-j} = \sum_{i=j+1}^{p-1} \frac{1}{p-i} \equiv -(H_{p-1} - H_j) \equiv H_j \pmod{p}. \quad (4.4)$$

Thus recalling that  $\alpha^{p-1} \equiv \beta^{p-1} \equiv 1 \pmod{p}$  now, we get

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (\alpha^{2k} - \beta^{2k}) H_{2k} \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (\beta^{2k} - \alpha^{2k}) H_{2k} \pmod{p},$$

i.e.,

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (\alpha^{2k} - \beta^{2k}) H_{2k} \equiv 0 \pmod{p}.$$

(ii) Let  $\alpha = 2 + \sqrt{3}$  and  $\beta = 2 - \sqrt{3}$ . Similarly, we also have

$$\begin{aligned} & \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^k (\alpha^{2k} + \beta^{2k}) H_{2k} \\ &= \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^{\frac{p-1}{2}-k} (\alpha^{p-1}\beta^{2k} + \beta^{p-1}\alpha^{2k}) H_{p-1-2k} \\ &\equiv - \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-1)^k (\beta^{2k} + \alpha^{2k}) H_{2k} \pmod{p}, \end{aligned}$$

which clearly implies (4.2).

(iii) Let  $\alpha = 3 + 2\sqrt{3}$  and  $\beta = 3 - 2\sqrt{3}$ . Since  $p \equiv \pm 1 \pmod{8}$ , similarly we can get

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (\alpha^{2k} - \beta^{2k}) H_{2k} \equiv 0 \pmod{p}.$$

Then (4.3) is immediately derived from (3.12).  $\square$

Now we are ready to prove (1.3), (1.4) and (1.7). It is not difficult to see that  $2^n P_n = U_n(4, -4)$  and  $P_{2n} = U_n(6, 1)$ . So by (1.6), (3.8) and (4.1),

$$\begin{aligned} & \sum_{k=0}^{\frac{p-1}{2}} \binom{p-1}{k} \binom{2k}{k}^2 \frac{P_k}{8^k} \\ &\equiv (-1)^{\frac{p-1}{2}} 16^{p-1} \left( \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 P_{2j} + 2p \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 P_{2j} H_{2j} \right) \equiv 0 \pmod{p^2}. \end{aligned}$$

(1.4) similarly follows from (3.9) and (4.2), since  $(-4)^n R_n = V_n(-16, 16)$  and  $(-1)^n R_{2n} = V_n(-14, 1)$ . Easily we can verify  $4^{n-1} W_n = U_n(32, -32)$  and  $2^{-n-2} W_{2n} = U_n(34, 1)$ . So (1.7) is also an easy consequence of (3.10) and (4.3).

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DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093,  
PEOPLE'S REPUBLIC OF CHINA

*E-mail address*: [mg1421007@smail.nju.edu.cn](mailto:mg1421007@smail.nju.edu.cn)

DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093,  
PEOPLE'S REPUBLIC OF CHINA

*E-mail address*: [haopan79@zoho.com](mailto:haopan79@zoho.com)